Exact Periodic Solitary Wave and Double Periodic Wave Solutions for the (2+1)-Dimensional Korteweg-de Vries Equation*

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A new technique, the extended ansatz function method, is proposed to seek periodic solitary wave solutions of integrable systems. Exact periodic solitary wave solutions for the (2+1)-dimensional Korteweg-de Vries (KdV) equation are obtained by using this technique. By using the trial function method, Jacobi elliptic function double periodic solutions are also constructed for this equation. This result shows that there exist periodic solitary waves in the different directions for the (2+1)-dimensional KdV equation.

Key words: Two-Dimensional KdV Equation; Extended Ansatz Function Method; Periodic Solitary Wave Solution; Double Periodic Solution.

1. Introduction

Boiti et al. derived the (2+1)-dimensional KdV equation [1]

$$u_t + 3(uv)_x + u_{xxx} = 0, \quad u_x - v_y = 0 \tag{1}$$

by using the idea of the weak Lax pair. (1) is also called Boiti-Leon-Manna-Pempinelli equation [2]. For $v = u$ and $y = x$, (1) reads as the ubiquitous KdV equation in dimensionless variables

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2}$$

Thus, (1) is a generalisation of the KdV equation. The rich dromion structures and localized structures are revealed by Lou et al. for (1) [3,4] and Wazwaz [5]. In the recent years, many authors have applied the Auto-Bäcklund transformation method [6] and the F-expansion method [7] and found abundant exact solutions for many nonlinear partial differential equations. But these obtained structures do not include some periodic solitary wave solutions which is a new type of soliton solutions.

Recently, Dai et al. introduced a new technique, the extended homoclinic test technique, which is used for seeking periodic solitary wave solutions of integrable equations and obtained periodic solitary wave solutions for integrable equations [8 – 12]. In this paper, we extend the technique mentioned above, introduce a new ansatz function $f(x,y,t) = b_1 e^{(Px+Qy+Wt)} + b_2 \cos(Kx+Ly+Vt) + b_3 e^{-(Px+Qy+Wt)}$, which is different from the function $f(x,y,t) = 1 + a_1 p(x,t) + a_2 q(y,t) + Ap(x,t)q(y,t)$ in [4], and consider (1). Then, we use the trial function method [13] to look for the double periodic wave solutions of (1). Exact periodic solitary wave and double periodic wave solutions are obtained for (1). This result shows that there exists periodic solitary waves in the different directions for (1). To our knowledge, some periodic solitary wave and double periodic wave solutions of the (2+1)-dimensional KdV equation are new solutions up to now.

The rest of this paper is organized as follows: in Section 2, a procedure for solving the (2+1)-dimensional KdV equation is given. Finally, some conclusions follow.

2. Exact Periodic Solitary Wave and Double Periodic Wave Solutions for the (2+1)-dimensional KdV Equation

Method 1.

We transform (1) into the bilinear form

$$D_y(D_t + D_x^3) f \cdot f = 0 \tag{3}$$

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by the transformation

$$u = 2\ln(fx,y,t))_{xy}, v = 2\ln(f(x,y,t))_{xx}, \tag{4}$$

where the bilinear operator $D_x^m D_t^n$ is defined as

$$D_x^m D_t^n a \cdot b = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \cdot a(x,t)b(x',t')|_{x=x',t=t'}. \tag{5}$$

Then (3) can be rewritten as

$$2\big(f_{yt}f - f_y f_t + f_{xxxy}f + 3f_{xx}f_{xy} - 3f_{xxy}f_x - f_{xxx}f_y\big) = 0. \tag{6}$$

We express the function f in the form

$$f(x,y,t) = b_1 \mathrm{e}^{(Px+Qy+Wt)} + b_2\cos(Kx+Ly+Vt) + b_3\mathrm{e}^{-(Px+QY+Wt)}. \tag{7}$$

Substituting (7) into (6), we obtain an algebraic equation for $\mathrm{e}^{j(Px+QY+Wt)}$, $\sin(Kx+Ly+vt)$, and $\cos(Kx+Ly+vt)$. Equating all therein coefficients to zero $(j=-1,0,1)$ yields a set of algebraic equations for $b_1, b_2, b_3, P, Q, W, K, L$ and V:

$$\begin{aligned}
&16b_1Qb_3P^3 + 4b_1Qb_3W - b_2^2LV + 4b_2^2LK^3 = 0,\\
&-b_2Lb_3P^3 + b_3Qb_2K^3 - b_3Qb_2V - b_2Lb_3W\\
&+3b_3Pb_2LK^2 - 3b_2Kb_3P^2Q = 0,\\
&-b_2b_3QW - b_3b_2VL + b_3P^3Qb_2 + b_2LK^3b_3\\
&-3b_2K^2b_3PQ - 3b_3P^2b_2LK = 0,\\
&-b_1Qb_2K^3 + b_2Lb_1P^3 + b_2Lb_1W + b_1Qb_2V\\
&-3b_1Pb_2LK^2 + 3b_2Kb_1P^2Q = 0,\\
&b_1P^3Qb_2 + b_2LK^3b_1 + b_2b_1QW - b_1b_2VL\\
&-3b_2K^2b_1PQ - 3b_1P^2b_2LK = 0.
\end{aligned} \tag{8}$$

Solving this system of algebraic equations with the aid of Maple, we can obtain the following results.

Case 1.

$$\begin{aligned}
&P = -\frac{1}{4}\frac{KLb_2^2}{b_1Qb_3}, \quad W = -\frac{1}{64}\frac{b_2^2LK^3(48b_1^2Q^2b_3^2 - L^2b_2^4)}{b_1^3Q^3b_3^3}, \quad V = \frac{1}{16}\frac{K^3(16b_1^2Q^2b_3^2 - 3L^2b_2^4)}{b_1^2Q^2b_3^2},\\
&K = K, \quad L = L, \quad Q = Q, \quad b_1 = b_1, \quad b_2 = b_2, \quad b_3 = b_3,
\end{aligned} \tag{9}$$

where K, L, Q, b_1, b_2, b_3 are free parameters. Setting $b_3 = \pm b_1$, we have

$$\begin{aligned}
f_1(x,y,t) &= 2b_1\cosh(Px+Qy+Wt) + b_2\cos(Kx+Ly+Vt),\\
f_2(x,y,t) &= 2b_1\sinh(Px+Qy+Wt) + b_2\cos(Kx+Ly+Vt).
\end{aligned} \tag{10}$$

Substituting (10) into (4) yields periodic soliton solutions of (1) as follows:

$$\begin{aligned}
u_1(x,y,t) &= 2\left\{\frac{4b_1^2PQ - b_2^2LK + 2b_1b_2(PQ-KL)\cosh(Px+Qy+Wt)\cos(Kx+Ly+Vt)}{(2b_1\cosh(Px+Qy+Wt) + b_2\cos(Kx+Ly+Vt))^2}\right.\\
&\quad\left. + \frac{2b_1b_2(PL+KQ)\sinh(Px+Qy+Wt)\sin(Kx+Ly+Vt)}{(2b_1\cosh(Px+Qy+Wt) + b_2\cos(Kx+Ly+Vt))^2}\right\},\\
v_1(x,y,t) &= 2\left\{\frac{4b_1^2P^2 - b_2^2K^2 + 2b_1b_2(P^2-K^2)\cosh(Px+Qy+Wt)\cos(Kx+Ly+Vt)}{(2b_1\cosh(Px+Qy+Wt) + b_2\cos(Kx+Ly+Vt))^2}\right.\\
&\quad\left. + \frac{4b_1b_2PK\sinh(Px+Qy+Wt)\sin(Kx+Ly+Vt)}{(2b_1\cosh(Px+Qy+Wt) + b_2\cos(Kx+Ly+Vt))^2}\right\},
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
u_2(x,y,t) &= 2\left\{\frac{-4b_1^2PQ - b_2^2LK + 2b_1b_2(PQ-KL)\sinh(Px+Qy+Wt)\cos(Kx+Ly+Vt)}{(2b_1\sinh(Px+Qy+Wt) + b_2\cos(Kx+Ly+Vt))^2}\right.\\
&\quad\left. + \frac{2b_1b_2(PL+KQ)\cosh(Px+Qy+Wt)\sin(Kx+Ly+Vt)}{(2b_1\sinh(Px+Qy+Wt) + b_2\cos(Kx+Ly+Vt))^2}\right\},
\end{aligned}$$

$$v_2(x,y,t) = 2\left\{\frac{-4b_1^2P^2 - b_2^2K^2 + 2b_1b_2(P^2-K^2)\sinh(Px+Qy+Wt)\cos(Kx+Ly+Vt)}{(2b_1\sinh(Px+Qy+Wt)+b_2\cos(Kx+Ly+Vt))^2} + \frac{4b_1b_2PK\cosh(Px+Qy+Wt)\sin(Kx+Ly+Vt)}{(2b_1\sinh(Px+Qy+Wt)+b_2\cos(Kx+Ly+Vt))^2}\right\}, \quad (12)$$

where all parameters are defined by (9).

Especially, when $L=0$, (11) and (12) become

$$u_1(x,y,t) = \frac{4b_1b_2KQ\sin(Kx+K^3t)\sinh(Qy)}{(2b_1\cosh(Qy)+b_2\cos(Kx+K^3t))^2},$$
$$v_1(x,y,t) = \frac{-2b_2K^2(2b_1\cos(Kx+K^3t)\cosh(Qy)+b_2)}{(2b_1\cosh(Qy)+b_2\cos(Kx+K^3t))^2}, \quad (13)$$

and

$$u_2(x,y,t) = \frac{4b_1b_2KQ\sin(Kx+K^3t)\cosh(Qy)}{(2b_1\sinh(Qy)+b_2\cos(Kx+K^3t))^2},$$
$$v_2(x,y,t) = \frac{-2b_2K^2(2b_1\cos(Kx+K^3t)\sinh(Qy)+b_2)}{(2b_1\sinh(Qy)+b_2\cos(Kx+K^3t))^2}, \quad (14)$$

where b_1, b_2, K and Q are free parameters.

Case 2.

$$L=L,\quad P=P,\quad W=-P^3,\quad V=0,\quad Q=0,$$
$$K=0,\quad b_1=b_1,\quad b_2=b_2,\quad b_3=b_3. \quad (15)$$

Similar as previous case, we set $b_3=\pm b_1$ and obtain the following periodic soliton solutions of (1):

$$u_1(x,y,t) = \frac{4b_1b_2PL\sinh(Px-P^3t)\sin(Ly)}{(2b_1\cosh(Px-P^3t)+b_2\cos(Ly))^2},$$
$$v_1(x,y,t) = \frac{4b_1P^2(2b_1+b_2\cosh(Px-P^3t)\cos(Ly))}{(2b_1\cosh(Px-P^3t)+b_2\cos(Ly))^2}, \quad (16)$$

and

$$u_2(x,y,t) = \frac{4b_1b_2PL\cosh(Px-P^3t)\sin(Ly)}{(2b_1\sinh(Px-P^3t)+b_2\cos(Ly))^2},$$
$$v_2(x,y,t) = \frac{-4b_1P^2(2b_1-b_2\cosh(Px-P^3t)\cos(Ly))}{(2b_1\sinh(Px-P^3t)+b_2\cos(Ly))^2}, \quad (17)$$

where b_1, b_2, P and L are free parameters.

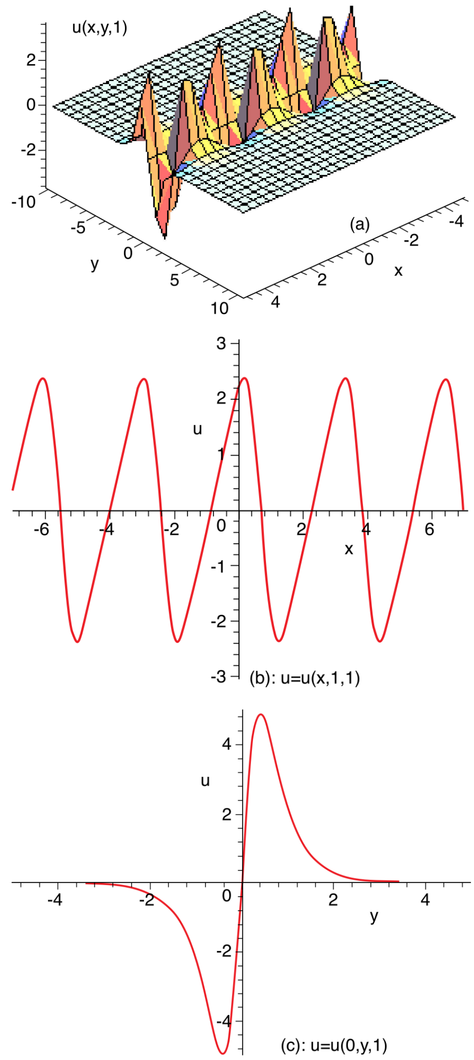

Fig. 1. Periodic soliton solution of (13) with $b_1=1$, $b_2=2$, $K=2$, $Q=2$; (a): $t=1$; (b): $y=1$, $t=1$; (c): $x=0$, $t=1$.

Case 3.

$$K=\sqrt{3}P,\quad w=8P^3,\quad L=L,\quad P=P,$$
$$Q=-\frac{\sqrt{3}L}{4b_3b_1}b_2^2,\quad v=0,\quad b_1=b_1, \quad (18)$$
$$b_2=b_2,\quad b_3=b_3.$$

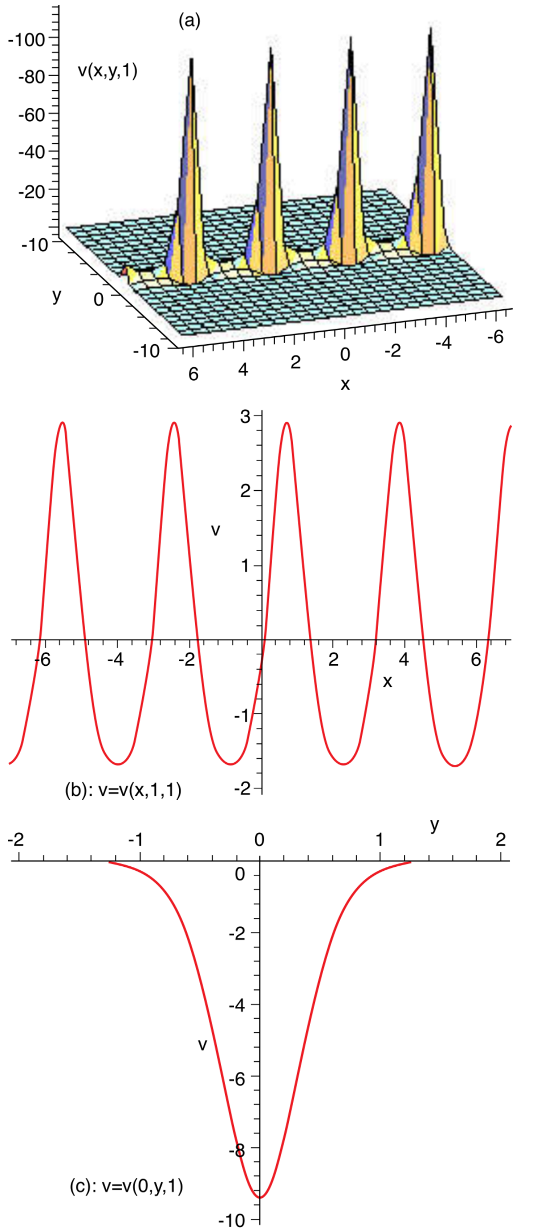

Fig. 2. Periodic soliton solution of (13) with $b_1 = 1$, $b_2 = 2$, $K = 2$, $Q = 2$; (a): $t = 1$; (b): $y = 1$, $t = 1$; (c): $x = 0$, $t = 1$.

Setting $b_3 = \pm b_1 = b_2$, we get

$$\begin{aligned} f_1(x,y,t) &= b_2\big[2\cosh(Px - \frac{1}{4}L\sqrt{3}y + 8P^3t) \\ &\quad + \cos(\sqrt{3}Px + Ly)\big], \\ f_2(x,y,t) &= b_2\big[2\sinh(Px + \frac{1}{4}L\sqrt{3}y + 8P^3t) \\ &\quad + \cos(\sqrt{3}Px + Ly)\big]. \end{aligned} \tag{19}$$

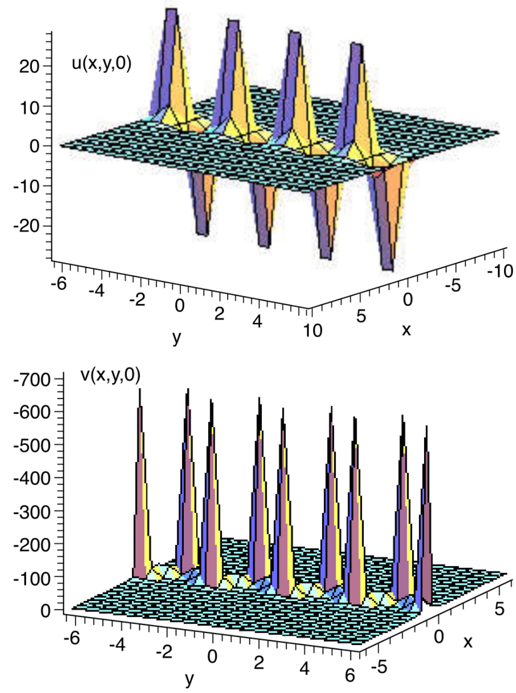

Fig. 3. Periodic soliton solution of (17) with $b_1 = 1$, $b_2 = 2$, $P = 2$, $L = 2$, and $t = 0$.

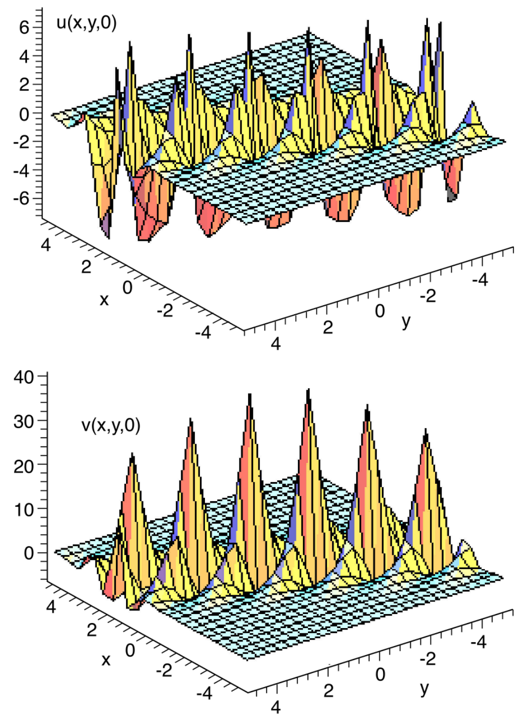

Fig. 4. Periodic soliton solution of (20) with $P = 2$, $L = 2$, and $t = 0$.

Substituting (19) into (4) yields periodic soliton solutions of (1) as follows:

$$
\begin{aligned}
u_1(x,y,t) &= -PL\left\{\frac{4\sqrt{3}+5\sqrt{3}\cosh(Px-\frac{1}{4}L\sqrt{3}y+8P^3t)\cos(\sqrt{3}Px+Ly)}{(2\cosh(Px-\frac{1}{4}L\sqrt{3}y+8P^3t)+\cos(\sqrt{3}Px+Ly))^2}\right.\\
&\qquad\left.-\frac{\sinh(Px-\frac{1}{4}L\sqrt{3}y+8P^3t)\sin(\sqrt{3}Px+Ly)}{(2\cosh(Px-\frac{1}{4}L\sqrt{3}y+8P^3t)+\cos(\sqrt{3}Px+Ly))^2}\right\},\\
v_1(x,y,t) &= 2P^2\left\{\frac{1-4\cosh(Px-\frac{1}{4}L\sqrt{3}y+8P^3t)\cos(\sqrt{3}Px+Ly)}{(2\cosh(Px-\frac{1}{4}L\sqrt{3}y+8P^3t)+\cos(\sqrt{3}Px+Ly))^2}\right.\\
&\qquad\left.+\frac{4\sqrt{3}\sinh(Px-\frac{1}{4}L\sqrt{3}y+8P^3t)\sin(\sqrt{3}Px+Ly)}{(2\cosh(Px-\frac{1}{4}L\sqrt{3}y+8P^3t)+\cos(\sqrt{3}Px+Ly))^2}\right\},
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
u_2(x,y,t) &= -PL\left\{\frac{4\sqrt{3}+3\sqrt{3}\sinh(Px+\frac{1}{4}L\sqrt{3}y+8P^3t)\cos(\sqrt{3}Px+Ly)}{(2\sinh(Px+\frac{1}{4}L\sqrt{3}y+8P^3t)+\cos(\sqrt{3}Px+Ly))^2}\right.\\
&\qquad\left.-\frac{7\cosh(Px+\frac{1}{4}L\sqrt{3}y+8P^3t)\sin(\sqrt{3}Px+Ly)}{(2\sinh(Px+\frac{1}{4}L\sqrt{3}y+8P^3t)+\cos(\sqrt{3}Px+Ly))^2}\right\},\\
v_2(x,y,t) &= 2P^2\left\{\frac{-7-4\sinh(Px+\frac{1}{4}L\sqrt{3}y+8P^3t)\cos(\sqrt{3}Px+Ly)}{(2\sinh(Px+\frac{1}{4}L\sqrt{3}y+8P^3t)+\cos(\sqrt{3}Px+Ly))^2}\right.\\
&\qquad\left.+\frac{4\sqrt{3}\cosh(Px+\frac{1}{4}L\sqrt{3}y+8P^3t)\sin(\sqrt{3}Px+Ly)}{(2\sinh(Px+\frac{1}{4}\sqrt{3}y+8P^3t)+\cos(\sqrt{3}Px+Ly))^2}\right\}.
\end{aligned}
\tag{21}
$$

As shown in Figure 4, the solutions $u_1(x,y,t)$ and $v_1(x,y,t)$ in (20) (similarly, other periodic soliton solutions) are obviously two periodic solitary waves which are periodic waves along the X-direction: $X=\sqrt{3}Px+Ly$ with period 2π, and are solitary waves along Y-direction: $Y=Px-\frac{1}{4}L\sqrt{3}y+8P^3t$. We note that here waves are two left-propagation waves along Y-direction: $Y=Px-\frac{1}{4}L\sqrt{3}y+8P^3t$. This is a interesting phenomenon to the evolution of shallow water waves. One problem, which we will study, is whether there exist a similar phenomenon to other equations of shallow water waves or not.

Method 2.

Suppose that (1) has the Jacobi elliptic function double periodic solution as follows:

$$
u=\frac{A+B\mathrm{cn}^2(\xi,m)}{C+\mathrm{cn}^2(\xi,m)},\quad v=\frac{E+F\mathrm{cn}^2(\xi,m)}{G+\mathrm{cn}^2(\xi,m)},
\tag{22}
$$

where $\xi=Px+Qy+Wt$ and A,B,C,E,F,G,P,Q,W are constants to be determined, m is the modulus of the Jacobi elliptic function $(0\le m\le 1)$. With the aid of Maple, we substitute (22) into (1) and balance the coefficients of $\mathrm{cn}(\xi,m)^j$ $(j=0,2,4,6,8)$ to yield a set of algebraic equations. By solving this set of equations, we can determine A,B,C,E,F,G,P,Q and W. Then, from (1) and (22) we obtain the double periodic function solutions of (1), which read

$$
u_1=-\frac{Q}{3}\frac{(3PF-4P^3+W+8m^2P^3)\mathrm{cn}^2(Px+Qy+Wt,m)-3EP}{P^2\mathrm{cn}^2(Px+Qy+Wt,m)},\quad v_1=\frac{E+F\mathrm{cn}^2(Px+Qy+Wt,m)}{\mathrm{cn}^2(Px+Qy+Wt,m)},
\tag{23}
$$

$$
\begin{aligned}
u_2 &= \frac{B\mathrm{cn}^2(Px+Qy+Wt,m)-B+2QP}{-1+\mathrm{cn}^2(Px+Qy+Wt,m)},\\
v_2 &= \frac{1}{3}\left[\frac{(4P^3Q-WQ-3P^2B+4P^3m^2Q)\mathrm{cn}^2(Px+Qy+Wt,m)}{QP(-1+\mathrm{cn}^2(Px+Qy+Wt,m))}+\frac{2P^3Q-4P^3m^2Q+WQ+3P^2B}{QP(-1+\mathrm{cn}^2(Px+Qy+Wt,m))}\right],
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
u_3 &= \frac{m^2B\mathrm{cn}^2(Px+Qy+Wt,m)+B-Bm^2-2Pm^2Q+2PQ}{1-m^2+m^2\mathrm{cn}^2(Px+Qy+Wt,m)},\\
v_3 &= \frac{1}{3}\left[\frac{(4m^4P^3Q-m^2WQ-3m^2P^2B-8P^3m^2Q)\mathrm{cn}^2(Px+Qy+Wt,m)}{QP(1-m^2+m^2\mathrm{cn}^2(Px+Qy+Wt,m))}\right.\\
&\quad\left.+\frac{WQm^2+6P^3m^2Q-2P^3Q-3P^2B+3P^2Bm^2-4P^3m^4Q-WQ}{QP(1-m^2+m^2\mathrm{cn}^2(Px+Qy+Wt,m))}\right],
\end{aligned}
\tag{25}
$$

where B, E, F, P, Q and W are free parameters.

3. Conclusion

In this paper, the (2+1)-dimensional KdV equation is investigated by the extended ansatz function method and trial function method. Some periodic solitary wave solutions and double periodic solutions are obtained for this equation. The result shows that there exists periodic solitary waves in the different directions for (2+1)-dimensional KdV equation. These methods are effective for seeking periodic solitary wave solutions and double periodic solutions of some integrable equations.

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